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Modified Adomian decomposition method for specific second order ordinary differential equations

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Abstract

In this paper, an efficient modification of Adomian decomposition method is introduced for solving second order ordinary differential equations. The proposed method can be applied to singular and nonsingular problems. The scheme is tested for some examples and the obtained results demonstrate efficiency of the proposed method.

Keywords: Adomian decomposition method; Singular and nonsingular ordinary differential equations

1. Introduction

In recent years, the studies of initial value problems in the second order ordinary differential equations (ODEs) have attracted the attention of many mathematicians and physicists. A large amount of literature developed concerning Adomian decomposition method [1–5,7], and the related modification [6,8,9,11] to investigate various scientific models. It is the purpose of this paper to introduce a new reliable modification of Adomian decomposition method. For this reason, a new differential operator is proposed which can be used for singular and nonsingular ODEs. In addition, the proposed method is tested for some examples and the obtained results show the advantage of using this method.

2. Modified Adomian decomposition method

Consider the initial value problem in the second order ordinary differential equation in the form

$$\begin{cases} y'' + P(x)y' + F(x,y) = g(x), \\ y(0) = A, y'(0) = B, \end{cases} \tag{1}$$

where $F(x,y)$ is a real function, $P(x)$ and $g(x)$ are given functions and A and B are constants. Here, we propose the new differential operator, as below

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$$L = \mathrm{e}^{-\int P(x)\mathrm{d}x} \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\int P(x)\mathrm{d}x} \frac{\mathrm{d}}{\mathrm{d}x}\right), \tag{2}$$

so, the problem (1) can be written as,

$$Ly = g(x) - F(x, y). \tag{3}$$

The inverse operator L^{-1} is therefore considered a two-fold integral operator, as below,

$$L^{-1}(\cdot) = \int_0^x \mathrm{e}^{-\int P(x)\,\mathrm{d}x} \int_0^x \mathrm{e}^{\int P(x)\,\mathrm{d}x}(\cdot)\,\mathrm{d}x\,\mathrm{d}x. \tag{4}$$

By operating L^{-1} on (3), we have

$$y(x) = \Phi(x) + L^{-1}g(x) - L^{-1}F(x, y), \tag{5}$$

such that

$$L\Phi(x) = 0. \tag{6}$$

The Adomian decomposition method introduce the solution $y(x)$ and the nonlinear function $F(x, y)$ by infinite series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{7}$$

and

$$F(x, y) = \sum_{n=0}^{\infty} A_n, \tag{8}$$

where the components $y_n(x)$ of the solution $y(x)$ will be determined recurrently. Specific algorithms were seen in [7,10] to formulate Adomian polynomials. The following algorithm:

$$\begin{cases} A_0 = F(u_0), \\ A_1 = F'(u_0)u_1, \\ A_2 = F'(u_0)u_2 + \frac{1}{2}F''(u_0)u_1^2, \\ A_3 = F'(u_0)u_3 + F''(u_0)u_1u_2 + \frac{1}{3!}F'''(u_0)u_1^3, \\ \vdots \end{cases} \tag{9}$$

can be used to construct Adomian polynomials, when $F(u)$ is a nonlinear function. By substituting (7) and (8) into (5),

$$\sum_{n=0}^{\infty} y_n = \Phi(x) + L^{-1}g(x) - L^{-1}\sum_{n=0}^{\infty} A_n. \tag{10}$$

Through using Adomian decomposition method, the components $y_n(x)$ can be determined as

$$\begin{cases} y_0 = \Phi(x) + L^{-1}g(x), \\ y_{n+1} = -L^{-1}A_n, \quad n \geqslant 0, \end{cases} \tag{11}$$

which gives

$$\begin{aligned} y_0 &= \Phi(x) + L^{-1}g(x), \\ y_1 &= -L^{-1}A_0, \\ y_2 &= -L^{-1}A_1, \\ y_3 &= -L^{-1}A_2, \\ &\vdots \end{aligned} \tag{12}$$

From (9) and (12), we can determine the components $y_n(x)$, and hence the series solution of $y(x)$ in (7) can be immediately obtained.

For numerical purposes, the n-term approximant

$$\Psi_n = \sum_{n=0}^{n-1} y_k,$$

can be used to approximate the exact solution.

Now, consider the Lane–Emden equation formulated as [4,11],

$$\begin{cases} y'' + \frac{2}{x}y' + F(x,y) = g(x), & 0 < x \leqslant 1, \\ y(0) = A, y'(0) = B, \end{cases} \tag{13}$$

where A and B are constants, $F(x,y)$ is a real function and $g(x) \in C[0,1]$ is given. Usually, the standard Adomian decomposition method is divergent to solve singular Lane–Emden equations. To overcome the singularity behavior, Wazwaz [11] defined the differential operator L in terms of two derivatives contained in the problem. He rewrote (13) in the form

$$Ly = -f(x,y) + g(x),$$

where the differential operator L is defined by

$$L = x^{-2}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\frac{\mathrm{d}}{\mathrm{d}x}\right).$$

Note that the above operator is a special kind of the proposed operator (2), since for Lane–Emden problem (13), $P(x)$ is equal to $\frac{2}{x}$, so,

$$\mathrm{e}^{-\int P(x)\,\mathrm{d}x} = x^{-2},$$

and

$$\mathrm{e}^{\int P(x)\,\mathrm{d}x} = x^2,$$

therefore we have

$$L = \mathrm{e}^{-\int P(x)\,\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\int P(x)\,\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}\right) = x^{-2}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\frac{\mathrm{d}}{\mathrm{d}x}\right),$$

and

$$L^{-1}(\cdot) = \int_0^x \mathrm{e}^{-\int P(x)\,\mathrm{d}x}\int_0^x \mathrm{e}^{\int P(x)\,\mathrm{d}x}(\cdot)\,\mathrm{d}x\,\mathrm{d}x = \int_0^x x^{-2}\int_0^x x^2(\cdot)\,\mathrm{d}x\,\mathrm{d}x.$$

3. Numerical examples

In this section, two initial ordinary differential equations are considered and then are solved by standard and modified Adomian decomposition methods.

Example 1. Consider the linear singular initial value problem,

$$\begin{cases} y'' + \frac{\cos x}{\sin x}y' = -2\cos x, \\ y(0) = 1, y'(0) = 0. \end{cases} \tag{14}$$

Standard Adomian decomposition method: we put

$$L(\cdot) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}(\cdot),$$

so

$$L^{-1}(\cdot) = \int_0^x \int_0^x (\cdot)\,\mathrm{d}x\,\mathrm{d}x.$$

In an operator form, Eq. (14) becomes

$$Ly = -\frac{\cos x}{\sin x} y' - 2\cos x. \tag{15}$$

By applying L^{-1} to both sides of (15) we have

$$y = y(0) + xy'(0) - L^{-1}\left(\frac{\cos x}{\sin x} y'\right) + L^{-1}(-2\cos x).$$

Proceeding as before we obtained the recursive relationship

$$\begin{cases} y_0 = y(0) + xy'(0) + L^{-1}(-2\cos x), \\ y_{n+1} = -L^{-1}\frac{\cos x}{\sin x} y'_n, \quad n \geqslant 0 \end{cases}$$

and the first few components are as follows:

$$\begin{cases} y_0 = 2\cos x - 1, \\ y_1 = -2\cos x + 2, \\ y_2 = 2\cos x - 2, \\ y_3 = -2\cos x + 2, \\ \vdots \end{cases}$$

It is easy to see that the standard Adomian decomposition method is divergent to solve this problem.

Modified Adomian decomposition method: According to (2), we put

$$L(\cdot) = \frac{1}{\sin(x)} \frac{\mathrm{d}}{\mathrm{d}x} \sin x \frac{\mathrm{d}}{\mathrm{d}x}(\cdot),$$

so

$$L^{-1}(\cdot) = \int_0^x \frac{1}{\sin x} \int_0^x \sin x(\cdot)\,\mathrm{d}x\,\mathrm{d}x.$$

In an operator form, Eq. (14) becomes

$$Ly = -2\cos x. \tag{16}$$

Now, by applying L^{-1} to both sides of (16) we have

$$L^{-1}Ly = -2\int_0^x \frac{1}{\sin x} \int_0^x \sin x(\cos(x))\,\mathrm{d}x\,\mathrm{d}x,$$

and it implies,

$$y(x) = y(0) + y'(0)x + \cos(x) - 1 \Rightarrow y(x) = \cos(x).$$

So, the exact solution is easily obtained by proposed Adomian method.

Example 2. Consider the linear nonsingular initial value problem,

$$\begin{cases} y'' + y' = 2x + 2, \\ y(0) = 0, \quad y'(0) = 0. \end{cases} \tag{17}$$

Standard Adomian decomposition method: we put

$$L(\cdot) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}(\cdot),$$

so

$$L^{-1}(\cdot) = \int_0^x \int_0^x (\cdot)\, dx\, dx.$$

In an operator form, Eq. (17) becomes

$$Ly = -y' + 2x + 2. \tag{18}$$

By applying $L^{-1}$to both sides of (18) we obtain

$$y = y(0) + xy'(0) - L^{-1}(y') + L^{-1}(2x + 2).$$

Proceeding as before we obtained the recursive relationship

$$\begin{cases} y_0 = y(0) + xy'(0) + L^{-1}(2x + 2), \\ y_{n+1} = -L^{-1}y'_n, \quad n \geqslant 0. \end{cases}$$

Thus, we have

$$\begin{cases} y_0 = \frac{1}{3}x^3 + x^2, \\ y_1 = -\frac{1}{12}x^4 - \frac{1}{3}x^3, \\ y_2 = \frac{1}{60}x^5 + \frac{1}{12}x^4, \\ y_3 = -\frac{1}{360}x^6 - \frac{1}{60}x^5, \\ y_4 = \frac{1}{2520}x^7 + \frac{1}{360}x^6 \\ \vdots \end{cases} \tag{19}$$

The noise terms (identical terms with opposite signs) appear in (19). Canceling these noises gives the exact solution,

$$y(x) = x^2.$$

Modified Adomian decomposition method: According to (2), we put

$$L = e^{-x}\frac{d}{dx}e^x\frac{d}{dx},$$

so

$$L^{-1}(\cdot) = \int_0^x e^{-x} \int_0^x e^x(\cdot)\, dx\, dx.$$

In an operator form, Eq. (17) becomes

$$Ly = 2x + 2. \tag{20}$$

Now, by applying L^{-1} to both sides of (20), we have

$$L^{-1}Ly = \int_0^x e^{-x} \int_0^x e^x(2x + 2)\, dx\, dx,$$

and it implies that

$$y(x) = y(0) + y'(0)x + x^2 \Rightarrow y(x) = x^2.$$

So, the exact solution is easily obtained by proposed Adomian method.

Example 3. Consider the nonlinear initial value problem [6],

$$\begin{cases} y'' + xy' + x^2y^3 = (2 + 6x^2)e^{x^2} + x^2e^{3x^2}, \\ y(0) = 1, \quad y'(0) = 0 \end{cases} \tag{21}$$

with the exact solution $y(x) = e^{x^2}$.

Standard Adomian decomposition method: we put

$$L(\cdot) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}(\cdot),$$

so

$$L^{-1}(\cdot) = \int_0^x \int_0^x (\cdot)\,\mathrm{d}x\,\mathrm{d}x.$$

According to (21) we have,

$$Ly = -xy' - x^2y^3 + g(x),$$

where $g(x) = (2+6x^2)\mathrm{e}^{x^2} + x^2\mathrm{e}^{3x^2}$.

Proceeding as before we obtain

$$\begin{cases} y_0 = y(0) + xy'(0) + L^{-1}(g(x)), \\ y_{n+1} = -L^{-1}(xy'_n) - L^{-1}(A_n), \quad n \geqslant 0 \end{cases}$$

when A_n's are Adomian polynomials of nonlinear term x^2y^3, as below [6]

$$\begin{cases} A_0 = x^2y_0^3, \\ A_1 = x^2(3y_0^2y_1), \\ A_2 = x^2(3y_0^2y_2 + 3y_0y_1^2), \\ A_3 = x^2(3y_0^2y_3 + 6y_0y_1y_2 + y_1^3), \\ \vdots \end{cases} \tag{22}$$

It must be noted that, to compute y_0, we use the Taylor series of $g(x)$ with order 10. In this case we obtain

$$\begin{cases} y_0 = 1 + x^2 + \frac{3}{4}x^4 + \frac{1}{3}x^6 + \cdots, \\ y_0 + y_1 = 1 + x^2 + \frac{1}{2}x^4 + \frac{2}{15}x^6 + \frac{1}{96}x^8 + \cdots, \\ y_0 + y_1 + y_2 = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{19}{420}x^8 + \cdots, \\ y_0 + y_1 + y_2 + y_3 = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8 + \frac{101}{12600}x^{10} + \cdots, \\ y_0 + y_1 + y_2 + y_3 + y_4 = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8 + \frac{1}{120}x^{10} + \frac{47}{33264}x^{12} + \cdots, \\ \vdots \end{cases}$$

Note that the Taylor series of the exact solution $y(x) = \mathrm{e}^{x^2}$ with order 10 is as below

$$\mathrm{e}^{x^2} = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8 + \frac{1}{120}x^{10} + O(x^{11}).$$

Modified Adomian decomposition method: According to (2), we put

$$L = \mathrm{e}^{-\frac{x^2}{2}}\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{\frac{x^2}{2}}\frac{\mathrm{d}}{\mathrm{d}x},$$

so

$$L^{-1}(\cdot) = \int_0^x \mathrm{e}^{-\frac{x^2}{2}} \int_0^x \mathrm{e}^{\frac{x^2}{2}}(\cdot)\,\mathrm{d}x\,\mathrm{d}x,$$

and we have,

$$\begin{cases} y_0 = y(0) + xy'(0) + L^{-1}(g(x)), \\ y_{n+1} = -L^{-1}(A_n), \quad n \geqslant 0. \end{cases}$$

By using Taylor series of $g(x), \mathrm{e}^{-\frac{x^2}{2}}$ and $\mathrm{e}^{\frac{x^2}{2}}$ with order 10 and Adomian polynomials mentioned in (22), we obtain,

$$\begin{cases} y_0 = 1 + x^2 + \frac{7}{12}x^4 + \frac{23}{96}x^6 + \cdots, \\ y_0 + y_1 = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{25}{672}x^8 + \cdots, \\ y_0 + y_1 + y_2 = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8 + \frac{1}{120}x^{10} + \frac{731}{443520}x^{12} + \cdots, \\ \vdots \end{cases}$$

So, the rate of convergence of modified Adomian is faster than standard Adomian method for this problem. The comparison between the results mentioned in Examples 1–3 show the power of the proposed method of this paper for these second order differential equations.

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